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On the Geometry of a Nodal Circular Cubic.

BY F. MORLEY.

In Crelle Bd. V the circular cubic with double focus on itself is treated by Schröter and Durège. I here give a geometrical account of the case when the curve, in addition, is nodal. It is then (Humbert, *American Journal* X, 3, p. 279) Quetelet's "focale à nœud," but I have not been able to find Quetelet's work. Some properties of the special case when the inflexion is at infinity are given by Booth (Quarterly Journal, Vol. III) under the name of the logocyclic curve.

§1. Let two tangents OP, OQ be drawn to a conic U, and let a conic U' through OPQ meet U again at P_1Q_1. Then we know that the tangents at P_1Q_1 meet at a point O_1 on U'. In proof, project P_1Q_1 to the circular points on the line ∞; then we have tangents OP, OQ to a circle U, and the circle OPQ or U' obviously goes through the centre O_1 of this circle. Also OO_1 is conjugate to PQ and P_1Q_1 with regard to U'. We shall need the following Lemmas from the geometry of the circle:

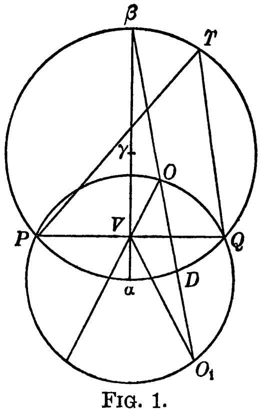

FIG. 1.

(1). Let OO_1, PQ be conjugate chords of a circle α, then the lines joining the ends of one to the centre of the other make with it equal angles. For let β be the pole of PQ. The centres of all chords through β lie on the circle γ whose diameter is $\alpha\beta$; hence if D be the centre of OO_1, the angle $PD\beta =$ angle $QD\beta$; also each is half angle $P\alpha Q$ and hence $= PO_1Q$.

(2). Since OV, O_1V are equally inclined to PQ at its centre (Lemma 1), they meet the circle in points equidistant from P, Q; hence angle $POV =$ angle QOD.

We may notice the corollaries that (1) the lines bisecting the angles POQ,

PO_1Q also bisect VOO_1, VO_1O, and hence by Lemma 1 meet on PQ, so that $PO:OQ = PO_1:O_1Q$; whence in a cyclic quadrilateral whose diagonals are conjugate, the rectangles of opposite sides are equal; (2) $OV.O_1V = PV^2$.

Returning to the conic U, let U' be a circle. We have the common chords PQ, P_1Q_1 of the circle and conic equally inclined to either axis, and therefore the radii CO, CO_1 equally inclined to either axis. Also from Lemma 2, angle $COP =$ angle O_1OQ, and therefore $COf_1 = O_1Of$. Hence the point O_1 has reference only to O, f, f_1, and is the same for all confocal conics when O is fixed; so that if we draw tangents from a fixed point O to a confocal family, the circles through O and the points of contact have a common radical axis (a result given in Wolstenholme, Problem 1079). The point O_1 clearly lies on the special circle Off_1, and also on the special circle (orthogonal to OFF_1) through O and the imaginary foci GG_1. We have of course $CO.CO_1 = Cf^2$.

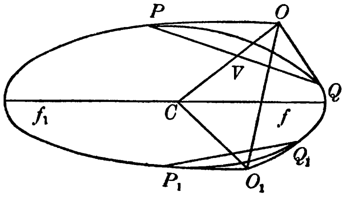

FIG. 2.

We know that the locus of P or Q is a circular cubic having a node at O, the nodal tangents bisecting the angle fOf_1; and that the cubic is also the locus of the feet of normals from O to the conics, and of feet of perpendiculars from O on its polars with regard to the conics. See a paper in the Messenger of Mathematics, April 1887, where I have given some of the geometry of this cubic, showing in particular that the tangents to it at P, Q meet at a point T on the cubic, so that PQ are "corresponding points"; that OP, OQ subtend equal or supplementary angles at any point of the curve (the fundamental property); in particular, that they make equal angles with either nodal tangent; that if PQ meets the cubic again at R, OR is perpendicular to PQ, and T, R are corresponding points, and that O is the centre of a circle touching TP, TQ, PQ (the latter of course at R). The angles which OP, OQ subtend are equal when R is between PQ, otherwise supplementary.

From what precedes, all circles through the node O and two corresponding points PQ pass through a fixed point O_1; and D being the centre of PO_1, OP, OQ subtend equal angles at D. Hence, from the fundamental property of the cubic, D is on the curve (Fig. 3). The line OV which bisects the lines joining corresponding points is parallel to the asymptote, for it is normal to the circle of infinitely large radius which is a limiting form of the confocal ellipses, i. e. it goes through the point at ∞ on the curve, K suppose. Now OV, OD make equal angles with a nodal tangent (Lemma 2); hence D, K correspond. But IJ being the circular points, the line at ∞ KIJ is perpendicular to any line and hence to VOK. Hence IJ correspond, and the tangents at them meet on the curve at D; or D is the double focus.

Any circle whose centre is D will have double contact with the cubic at IJ, for DI, DJ are tangents to both curves.

Since D, K correspond, angle $DQO =$ angle KQO, and if QD meets the cubic again at Q_1, angle $DQ_1O =$ angle KQ_1O. Hence angle QOQ_1 is right, or any chord through the double focus subtends a right angle at the node.

Let the line joining any point H on the curve to PQ meet the curve again at pq (see Fig. 6, where, however, H is special). Then, since angles OHP, OHQ are equal or supplementary, angles OHp, OHq are equal or supplementary, and therefore p, q also correspond. Conversely, the lines joining P, Q to p, q meet in two points on the curve which also correspond (see Salmon, Plane Curves, p. 132). The point O being equidistant from the lines joining corresponding points to any point on the curve (fundamental property) must be equidistant from the four lines Pp, Pq, Qp, Qq; or the four lines joining one pair of points to another pair touch a circle whose centre is the node, and intersect again on the curve (compare the paper above referred to).

In particular, QDQ_1 and PK (Fig. 3) meet on the curve, so that PQ_1 is parallel to the asymptote. Hence OV, which bisects PQ, also bisects QQ_1, or the locus of centres of chords through the double focus is the line through the node parallel to the asymptote.

The angle $PDQ =$ twice angle PO_1Q (Lemma 1) $= 360° -$ twice angle POQ. But O being centre of circle inscribed in TPQ, $POQ = 90° + \frac{1}{2} PTQ$. Hence $PDQ = 180° - PTQ$, or the circle circumscribed to the triangle formed by two tangents from a point on the curve, and the chord of contact, goes through the double focus. It is clear (see Fig. 1) that the tangents to the circle OPQ at P, Q meet on the circle TPQ. The locus of the centre of this latter circle may be seen to be a hyperbola of no apparent interest.

Let HH_1 be the chord through D perpendicular to OO_1. Since HH_1 is perpendicular to OD, HH_1 correspond. Hence OH, OH_1 make equal angles with the nodal tangents, and since OH, OH_1 are at right angles (HH_1 passing through D), they bisect the angles between the nodal tangents. Also, HH_1 being perpendicular to OD, the tangents at HH_1 are parallel to the asymptote. These points are of great use later.

Since D, K correspond, the tangent at D meets the asymptote on the curve, at A suppose. Since OD, OK subtend equal angles at D, the tangent and the asymptote are equally inclined to OD. Also AOD is a right angle since AD is a chord through the double focus. Hence if E is the centre of HH_1, $AODE$ is a rectangle.

§2. In connection with what follows see Salmon's Plane Curves, §§278–281. The bicircular quartic with a finite node being the pedal of a conic and the inverse of another conic, a circular cubic is the pedal of a parabola and the inverse of a conic with regard to a point on it, and the cubic in question having perpendicular nodal tangents, is the pedal of a parabola with regard to a point on the directrix and the inverse of a rectangular hyperbola with regard to a point on itself.

Since V is on a fixed line and PQ makes equal angles with OV, O_1V, we see that PQ envelopes a parabola with focus O_1 and directrix OV; and OR is

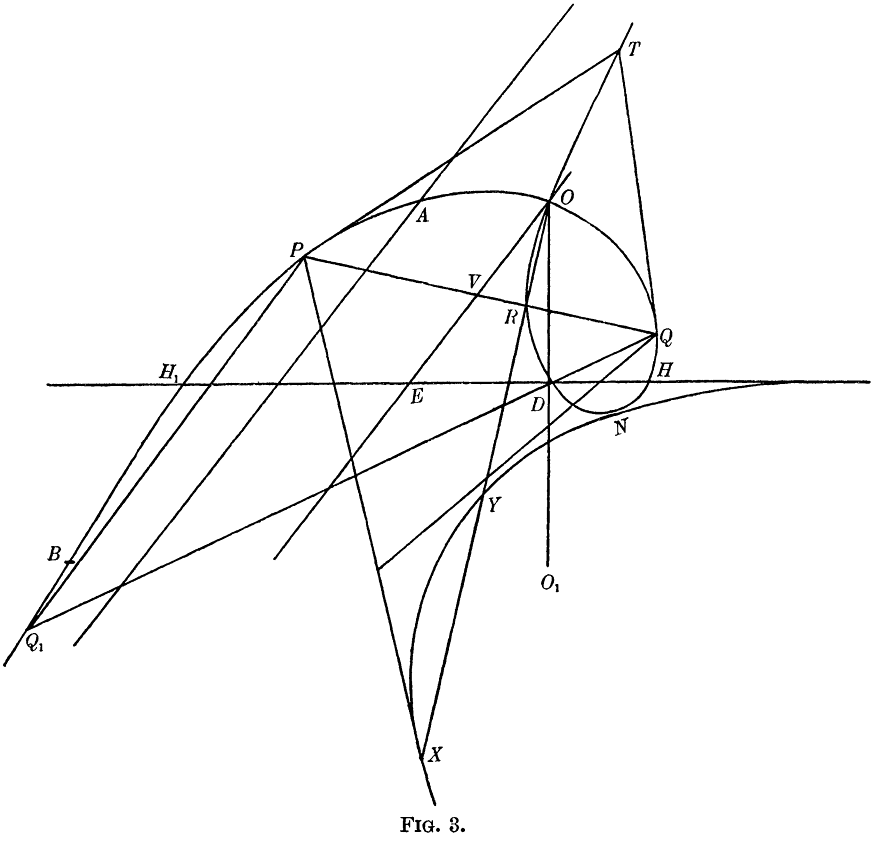

FIG. 3.

perpendicular to PQ. Thus we see *ab initio* that the curve is the pedal of a parabola. That it is the inverse of a hyperbola follows by reciprocation, or may be obtained directly from the fundamental property.

We now show that the tangents to the parabola from P, Q (which are perpendicular to OP, OQ) touch the parabola where OR meets it. It is enough to show that if PQ, PX be tangents to a parabola, the perpendiculars from X to PQ and from P to PX meet on the directrix. But these are perpendiculars of the triangle formed by PQ, PX and the tangent consecutive to PX. And we know that the orthocentre of any tangent triangle is on the directrix. It is obvious that PX, QY intersect on the polar of O, i. e. the line O_1B parallel to HH_1, and also on the circle $OPQO_1$. When QR coincide (at N in fig.), since the tangents at P, Q meet on the cubic and PQ is the tangent at Q, P must be the inflexion, and since it is the intersection of tangents from P, Q to the parabola, it must coincide with B, where the polar of O meets the cubic. Hence the inflexion and node are equidistant from HH_1. The curves clearly touch at N, and ON is the common normal. Since B, N correspond, OV bisects BN. Hence the tangent from the inflexion to the cubic is bisected by the parallel to the asymptote through the node.

If p, q correspond, then since OR, PQ make equal angles with Rp, Rq, they divide pq harmonically. Making pq coincide with PQ, the line PRQ is divided harmonically at the contact with the parabola.

It is worth while to call attention to the fact that D bisects OO_1. It may be proved (but is apart from the present purpose) that the four double foci of a nodal bicircular quartic bisect the lines joining the node to the foci of the conic which is the negative pedal of the quartic.

§3. We will now obtain the position of the four single foci of the cubic, by considering the cubic as the inverse of a rectangular hyperbola with regard to a point O on it.

The tangent at O inverts into a parallel to the finite asymptote. The parallels to the asymptotes through O become the nodal tangents. The four foci of the hyperbola become the four foci of the cubic. The following consideration may aid in clearing up this last point, which is usually stated without proof. Let O be the origin, and let a point F, whose coordinates are a, 0, be a focus of a curve. Let F_1G_1 be antipoints of O, F; the coordinates of F_1 are $a/2$, $\iota a/2$.

The circle OFF_1 is $x^2 + y^2 - ax - \iota ay$, or $(x + \iota y)(x - \iota y - a) = 0$, i. e. it breaks up into the imaginary lines OF_1I, F_1FJ. Since the latter is a tangent to the curve, so also is the circle. Since $OF_1^2 = 0 = OG_1^2$, when we invert the antipoints and the circular points are interchanged, the circle becomes a tangent through a circular point, and F remains a focus of the inverse curve.

Since the tangents at corresponding points of a curve and its inverse make equal angles with the radius vector, the images of O with regard to the axes will become the points HH_1 at which the tangents are parallel to the asymptote.

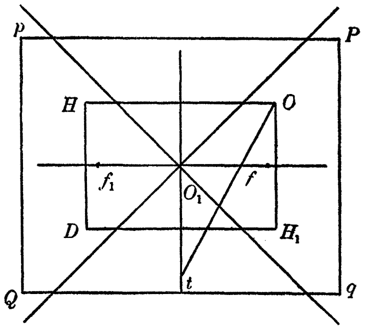

FIG. 4.

The end of the diameter through O lies on the circle OHH_1 and hence becomes the point where HH_1 cuts the cubic, that is the double focus. The ends of any diameter become corresponding points.

The foci ff_1 of the hyperbola are on the axis ff_1, which inverts into a circle through O with centre H_1. They are also on the circle Off_1H. If the tangent at O meets the conjugate axis at t, then the angle which the circle makes with $Ot =$ angle $tHO =$ angle tOH; hence in the cubic the foci lie on the line through H which makes with the asymptote the same angle as OH, and are the points where this line is cut by the circle through O with centre H_1.

Interchanging H and H_1, we have the imaginary foci, the antipoints of the real foci.

Since (Fig. 5) H is the pole of OO_1 with regard to the circle whose centre is H_1 and radius HO or HO_1, we see that OO_1 and ff_1 are conjugate chords. Hence DOf, f_1DO are similar triangles, and (cor. 2 to Lemma 2 at the beginning) $OD^2 = Df.Df_1$, a result generalized later.

It is clear that the focal circle with centre H_1 and radius H_1O, being the inverse of an axis, cuts the cubic orthogonally at points where the curvature is stationary.

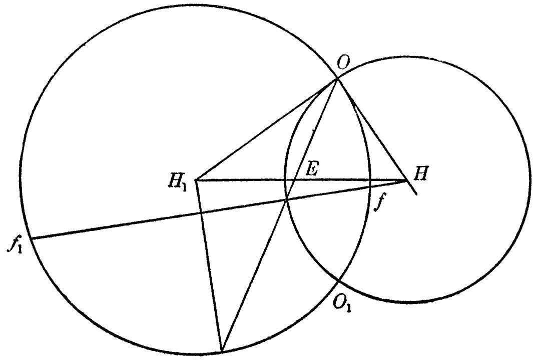

FIG. 5.

In the hyperbola we have $fP - f_1P = \pm k$, — when P is on the same branch as O. Hence on inversion

$$\frac{fP}{Of.OP} - \frac{f_1P}{Of_1.OP} = \pm k,$$

$+$ for the loop, because the branch of the hyperbola away from O inverts into the loop; f_1 is now the focus inside the loop. To determine k, take P at H_1 where $H_1f = H_1f_1 = H_1O$; then

$$\frac{1}{Of} - \frac{1}{Of_1} = -k,$$

and hence

$$\frac{fP \pm OP}{Of} = \frac{f_1P \pm OP}{Of_1},$$

where for points on the loop we take the $+$ sign.

Again in the hyperbola we have $Pf.Pf_1 = PO_1^2$, $Of.Of_1 = OO_1^2$, and hence on inversion

$$\frac{Pf}{OP.Of}\cdot\frac{Pf_1}{OP.Of_1} = \left(\frac{PO_1}{OP.OO_1}\right)^2, \quad Of.Of_1 = OO_1^2,$$

whence

$$Pf.Pf_1 = PO_1^2,$$

so that the cubic is the locus of a point whose distances from 3 fixed points are connected by the relation $r_1r_2 = r_3^2$. It is to be noticed that O_1 is not a focus. In the linear relation above connecting the distances from f, f_1, O, we may regard O as an improper focus, for the line from it to a circular point fulfils the condition for a tangent.

From the general case, when we take any conic and a point off it, we get that the product of focal distances of P is in a constant ratio to the rectangle of segments of a chord from P to a fixed circle, namely, the inverse of the director circle. The ratio becomes an equality when the fixed point is on the conic, in which case the quartic becomes a cubic.

From the equation $Pf.Pf_1 = PO_1^2$ we get that the line of foci meets the cubic at points on the circle H_1DO_1 as well as at H.

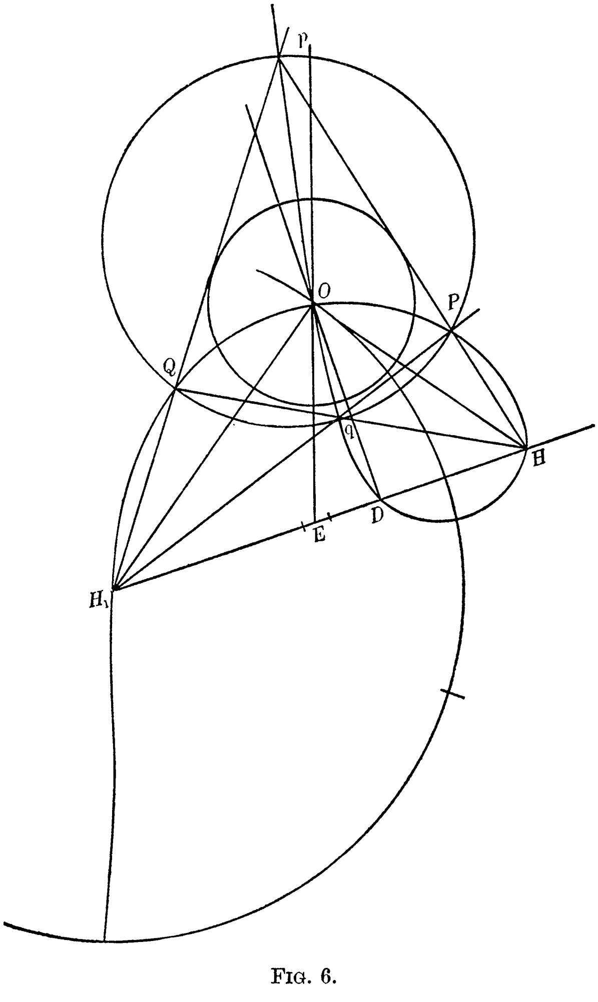

FIG. 6.

The cubic is its own inverse with regard to H or H_1 (Fig. 6). For let HPp be a chord through H. We have to show $HP.Hp = HO^2$. Inverting we have

(Fig. 4) a circle through OH, and Pp are clearly on a line parallel to ff_1. We have to show in the hyperbola

$$\frac{HP}{OH.OP}\cdot\frac{Hp}{OH.Op}=\frac{1}{HO^2},$$

or

$$HP.Hp=OP.Op,$$

which is obvious. This clearly applies to any nodal cubic.

Join Pp to H_1, cutting the cubic at qQ. We have shown above that PQ, pq (since they join corresponding points to a point on the curve) are corresponding points, that Qq goes through H, and that the four lines touch a circle with centre O. Also, since $HP.Hp=HQ.Hq=HO^2$, the four points lie on a circle orthogonal to the circles whose centres are H, H_1 and radii HO, H_1O; hence the centre of this circle is on OD, the radical axis of these circles. Since the circle OPQ has its centre on HH_1, it is orthogonal to the circle pPQ; the former belongs to a coaxial set through OO_1, the latter to the orthogonal coaxial set whose limiting points are OO_1. The line joining the centres of these circles, being perpendicular to PQ at its centre V, will touch the parabola of Fig. 3, for V is on the directrix. It therefore joins corresponding points, but they are imaginary, being in fact the antipoints of PQ. If it meets the cubic at r, clearly Or is a perpendicular on it, and since ROr is a right angle (Fig. 3), Rr goes through D and bisects OV. We thus have what is probably the simplest geometrical definition. Let O be a fixed point. Take Z on a fixed line through it, join Z to a fixed point D, and take R on ZD such that $ZR=ZO$. The locus of R is a right circular cubic of which O is node, OZ parallel to the asymptote, D double focus. This is the simplest starting-point for geometrical treatment.

Returning to Fig. 6, we may see by inversion that the diagonals PQ, pq meet on OD. This is, however, the known fact that when a quadrilateral is inscribed in one circle and described about another, the diagonals (and also the lines joining opposite points of contact) meet at a point on the line of centres. See Casey's "Sequel to Euclid," pp. 108, 94.

We see that the cubic is the locus of intersections of tangents from fixed points to circles with a given centre. The quadrilateral formed by the tangents is cyclic when the points subtend a right angle at the centre. For these circles with centre at the node, in connection with the theory of confocal conics, see the paper in the Messenger of Math. above referred to.

§4. The *raison d'être* of the above treatment of the *focale à nœud* is that any projective theorem for a nodal cubic or tricuspidal quartic becomes intuitive by projection or reciprocation. The leading feature of the geometry is the parabola of Fig. 3. Bearing in mind that the polar of the node with regard to this parabola will contain the two imaginary inflexions as well as the real one, we see that in a nodal cubic the envelope of the line joining corresponding points is a conic touching the cubic at the three sextactic points, and touching the nodal tangents at points on the line of inflexions, so that the node and line of inflexions are pole and polar with regard to it. It is the Cayleyan of the cubic of which the given cubic is the Hessian. Salmon, §177; and Durège, Curven dritter Ordnung, §540.

The following may be instanced as properties immediately following from the *focale à nœud:*

(1). Any two points on a nodal cubic and the points of contact of tangents from them lie on a conic. For the circle TPQ (Fig. 3) goes through D.

(2). Let PQ be corresponding points, T their tangential, R the point on the curve collinear with PQ, O the node. Any line through T, or any line joining corresponding points, is cut harmonically by OR, PQ, and the curve. If qTq_1, qp are such lines, pq_1 goes through R.

(3). If we draw tangents RX, RY, join XY to corresponding points xy, meeting the curve at corresponding points $\xi\eta$, then $PQxy\xi\eta$ lie on a conic, and a conic will pass through PQ and touch the lines joining XY to xy.

(4). The line OT is cut harmonically by the line of inflexions and the line PQ.

(5). A complete quadrilateral is inscribed in a cubic whose node is O. The 3 diagonals form a triangle such that the lines from O to the angles meet the sides on the cubic.

(6). Four lines meet at $AA_1BB_1CC_1$. Conics through AA_1BB_1 and AA_1CC_1 meet again at OO_1. With OO_1 as nodes, cubics are drawn through $AA_1BB_1CC_1$. Then they touch at AA_1, the tangents meet on the curve and on OO_1, at a point which with AA_1 cuts OO_1 harmonically; a conic through AA_1OO_1, with regard to which AA_1 and OO_1 are conjugate, will touch the cubics at AA_1.

(7). In the cardioid a tangent meets the curve at two points, the tangents at which meet the tangent at the vertex at points equidistant from the vertex.

Results in relation to some here considered are given by McIntosh (Educational Times Reprint, XXXVIII, p. 82) and Wolstenholme (Ib. XLII, p. 81, and XLIII, p. 77, and Problems, Nos. 1824, 1850, 1841).

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